Second order gravitational effects on CMB temperature anisotropy in Λ dominated flat universes

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We study second order gravitational effects of local inhomogeneities on the cosmic microwave background radiation in flat universes with matter and a cosmological constant Λ. We find that the general relativistic correction to the Newtonian approximation is negligible at second order provided that the size of the inhomogeneous region is sufficiently smaller than the horizon scale. For a spherically symmetric top-hat type quasi-linear perturbation, the first order temperature fluctuation corresponding to the linear integrated Sachs-Wolfe (ISW) effect is enhanced(suppressed) by the second order one for a compensated void(lump). As a function of redshift of the local inhomogeneity, the second order temperature fluctuations due to evolution of the gravitational potential have a peak before the matter-Λ equality epoch for a fixed comoving size and a density contrast. The second order gravitational effects from local quasi-linear inhomogeneities at a redshift $z \sim 1$ may significantly affect the cosmic microwave background.

I. INTRODUCTION

Generation of temperature anisotropy in the Cosmic Microwave Background (CMB) due to linear pertubatons of gravitational potentials is called the Sachs-Wolfe (SW) effect[1]. There are two contribution to the SW effect. The first one called the non-integrated Sachs-Wolfe effect is produced by fluctuations of gravitational potentials at the surface of last scattering. The second one called the integrated Sachs-Wolfe (ISW) effect is generated by time-varying gravitational potentials as the CMB photons pass through them from the recombination epoch to the present epoch.

Recently, much attention has been paid to the non-linear version of the ISW effect called the Rees-Sciama (RS) effect[2] on cosmological scales, since the RS effect of local inhomogeneities at quasi-linear regime may explain the origin of observed anomalies [3, 4, 5, 6, 7, 8] in the large-angle CMB anisotriopy [9, 10]. It is argued that the anisotropy for compensated asymptotically expanding local voids can be larger because the second order effect enhances the ISW effect [11]. For compensated local lumps, the nature of the second order effect at the Λ dominated epoch remains unknown.

So far, two types of treatment have been used for deriving the RS effect of local inhomogeneities. One is the general relativistic treatment [12, 13, 14], in which a non-linear version of the SW effect at second order has been studied. Another one is the simplified Newtonian treatment [15, 16, 17, 18, 19, 20, 21] in which influences of nonlinear halo clustering upon the CMB temperature fluctuations and gravitational lensing phenomena have been explored.

In this paper, we first study the RS effect comparatively using the general relativistic second order perturbation theory and the Newtonian approximation and we show the consistency between the two approaches in the perturbative regime. Next, we consider the RS effect of a compensating quasi-linear void/lump modelled by a spherically symmetric top-hat type density perturbation at a redshift z and we investigate the temporal change in the linear and second order temperature fluctuations as functions of z. In §2, we derive the solutions of Einstein equations for general relativistic perturbations at second order in the Poisson gauge (generalized longitudinal gauge) and consider the limit of κ ($\sim$ the scale of inhomogeneities / Hubble radius) $\ll 1$. In §3, we study the perturbations in cosmological Newtonian approximations and their relation to the general relativistic perturbations up to second order. In §4, we study the temperature anisotropy owing to first and second order perturbations based on relativistic perturbation theory [22]. In §5, we investigate the correlations between first order and second order temperature fluctuations and their temporal behavior for a spherical top-hat type density perturbation. §6 is dedicated to concluding remarks. In Appendices A and B, the main components of Einstein equations and the derivation of their solutions are shown. In Appendix C, the integrations of metric perturbations for spherically symmetric top-hat type perturbations along the light paths are shown.

II. GENERAL RELATIVISTIC SECOND ORDER PERTURBATIONS

In what follows, we use the units of $8\pi G = c = 1$, the Greek and Latin letters denote $0,1,2,3$ and $1,2,3$, respectively. Index "B" represents the value for the unperturbed background spacetime. $\delta_{ij}(= \delta^i_j = \delta^{ij})$ are the Kronecker delta, and subscripts (n) correspond to n-th order quantities.

As a function of conformal time $\tilde\eta$ and Cartesian coordinates $\tilde x^i$, the metric of spatially flat Friedmann-Robertson-Walker (FRW) universes with first order and second order perturbations $\psi^{(n)}, \phi^{(n)}, z_i^{(n)}, \chi_{ij}^{(n)}, n = 1, 2$ is described by

$$ds^2 \equiv g_{\mu\nu}d\tilde x^\mu d\tilde x^\nu = \tilde a^2(\tilde\eta)\Big\{ -(1 + 2\psi^{(1)} + \psi^{(2)})d\tilde\eta^2 + \Big[z_i^{(1)} + \frac{1}{2}z_i^{(2)}\Big]d\tilde\eta d\tilde x^i$$
$$+ \Big[(1 - 2\phi^{(1)} - \phi^{(2)})\delta_{ij} + \chi_{ij}^{(1)} + \frac{1}{2}\chi_{ij}^{(2)}\Big]d\tilde x^i d\tilde x^j\Big\}, \tag{2.1}$$

where $\tilde a(\tilde\eta)$ is the scale factor, $\tilde\eta$ $(= \tilde x^0)$ is related to the cosmic time t by $dt = \tilde a(\tilde\eta)d\tilde\eta$, and $\chi_{ij}^{(n)}$ satisfy $\chi_l^{(n)l} \equiv \delta^{lm}\chi_{lm}^{(n)} = 0$. In what follows, we use the normalized scale factor $a = \tilde a H_0^{-1}$, the comoving coordinate $x^i = \tilde x^i/H_0^{-1}$, and the conformal time $\eta = \tilde\eta/H_0^{-1}$ where H_0^{-1} is the Hubble radius at present η_0. The scale factors at present are defined as $\tilde a_0 = 1$ and $a_0 = H_0^{-1}$. Note that the scale fatcor a has a dimension of length while η and r do not have dimensions in our notation.

Energy density and 4-velocity of dust matter are written in terms of the background quantities and perturbations as

$$\rho = \rho_B(\eta) + \delta^{(1)}\rho + \frac{1}{2}\delta^{(2)}\rho, \tag{2.2}$$

$$u^\mu = \frac{1}{a}\Big[\delta^\mu_0 + v^{(1)\mu} + \frac{1}{2}v^{(2)\mu}\Big], \tag{2.3}$$

where v^i denotes the 3-velocity of the dust matter. From the condition $g_{\mu\nu}u^\mu u^\nu = -1$, we obtain the relations

$$v^{(1)0} = -\psi^{(1)}, \tag{2.4}$$

$$v^{(2)0} = -\psi^{(2)} + 3(\psi^{(1)})^2 + [2z_i^{(1)} + v_i^{(1)}]v^{(1)i}. \tag{2.5}$$

Einstein equations for the unperturbed background with dust matter and a cosmolological constant Λ are

$$3(a'/a)^2 = (\rho_B + \rho_\Lambda)a^2, \tag{2.6}$$

$$6(a'/a)' = -(\rho_B - 2\rho_\Lambda)a^2 \tag{2.7}$$

where ρ_Λ is the energy density of the cosmological constant Λ and a prime denotes derivative with respect to η.

To fix the gauge freedom of perturbations, we adopt the Poisson gauge defined by

$$\delta^{lm}z_{l,m}^{(n)} = 0 \quad \text{and} \quad \delta^{lm}\chi_{kl,m}^{(n)} = 0 \tag{2.8}$$

for $n = 1$ and 2. This gauge is a generalized version of the longitudinal gauge which is defined by $z_i^{(n)} = 0$ and $\chi_{ij}^{(n)} = 0$ [23], and gives us a metric expression convenient for a Newtonian interpretation, as well as the longitudinal gauge [24].

In what follows, we consider only scalar-type perturabtions at linear order. Then, we have

$$z_l^{(1)} = 0 \quad \text{and} \quad \chi_{kl}^{(1)} = 0 \tag{2.9}$$

The Ricci tensor, the Einstein tensor and the energy-momentum tensor for dust matter are shown in Appendix A. Solving the Einstein equation we obtain the expression of first order scalar-type perturbations in the growing

mode in terms of functions $P(\eta)$ and $F(\mathbf{x})$ as[1]

$$\psi^{(1)} = \phi^{(1)} = -\frac{1}{2}\Big(1 - \frac{a'}{a}P'\Big)F, \tag{2.10}$$

$$z_i^{(1)} = 0, \quad \chi_{ij}^{(1)} = 0, \tag{2.11}$$

$$\delta\rho^{(1)}/\rho_B = \frac{1}{\rho_B a^2}[(a'/a)P' - 1]\Delta F + \frac{3}{2}(a'/a)P'F, \tag{2.12}$$

$$v^{(1)0} = -\frac{1}{2}[(a'/a)P' - 1]F, \quad v^{(1)i} = \frac{1}{2}P'F_{,i}, \tag{2.13}$$

where $F_{,i}$ is $\partial F/\partial x^i$, Δ is the Laplacian $\partial^2/\partial x^i \partial x^i$, and

$$
\begin{aligned}
P(\eta) &= -\frac{2}{3\Omega_{m0}}\tilde{a}^{-3/2}[\Omega_{m0} + \Omega_{\Lambda 0}\tilde{a}^3]^{1/2}\int_0^{\tilde{a}} d\tilde{a}\tilde{a}^{3/2}[\Omega_{m0} + \Omega_{\Lambda 0}\tilde{a}^3]^{-1/2} + \frac{2}{3\Omega_{m0}}\tilde{a}, \\
\eta &= \int_0^{\tilde{a}} d\tilde{a}\tilde{a}^{-1/2}[\Omega_{m0} + \Omega_{\Lambda 0}\tilde{a}^3]^{-1/2},
\end{aligned}
\tag{2.14}
$$

where Ω_{m0} and $\Omega_{\Lambda 0}$ are the density parameters for matter and the cosmological constant at present. $P(\eta)$ is the solution of the growing mode in equation

$$P'' + \frac{2a'}{a}P' - 1 = 0. \tag{2.15}$$

Note that the potential function $F(\mathbf{x})$ is related to the first order matter density contrast ϵ_m in comoving slices (defined in the comoving synchronous gauge) as

$$\epsilon_m = \frac{1}{\rho_B a^2}[(a'/a)P' - 1]\Delta F. \tag{2.16}$$

Next we consider relativistic second order perturbations corresponding to the first order perturbations in the growing mode. From Eqs. (B9) - (B12) in Appendix B, which are derived by solving the Einstein equations $\delta_2 G_i^j = \delta_2 T_i^j$, we have

$$\phi^{(2)} = \zeta_1 F_{,l}F_{,l} + \zeta_2 \cdot 100\Psi_0 + \zeta_3 F^2 + \zeta_4 \cdot 100\Theta_0, \tag{2.17}$$

$$\psi^{(2)} = \xi_1 F_{,l}F_{,l} + \xi_2 \cdot 100\Psi_0 + \xi_3 F^2 + \xi_4 \cdot 100\Theta_0, \tag{2.18}$$

where

$$
\begin{aligned}
\zeta_1 &= \frac{1}{4}P\Big(1 - \frac{a'}{a}P'\Big), \qquad \zeta_2 = \Big\{\frac{1}{21}\frac{a'}{a}\Big(PP' - \frac{1}{6}Q'\Big) - \frac{1}{18}\Big[P + \frac{1}{2}(P')^2\Big]\Big\}, \\
\zeta_3 &= \frac{1}{4}P'\Big\{\frac{a'}{a} + \Big[-\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\Big]P'\Big\}, \qquad \zeta_4 = -\frac{1}{3}\frac{a'}{a}P'\Big(1 - \frac{a'}{a}P'\Big),
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
\xi_1 &= \zeta_1, \qquad \xi_2 = \zeta_2, \\
\xi_3 &= \frac{1}{4}\Big\{4 - 7\frac{a'}{a}P' + \Big[-\frac{a''}{a} + 5\Big(\frac{a'}{a}\Big)^2\Big](P')^2\Big\}, \qquad \xi_4 = \frac{1}{6}\Big\{2 - \frac{6a'}{a}P' + \Big[-\frac{2a''}{a} + 8\Big(\frac{a'}{a}\Big)^2\Big](P')^2\Big\},
\end{aligned}
\tag{2.20}
$$

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[1] In paper [12] (referred as Paper[12]), the first order perturbations in the Poisson gauge were derived by transforming the solution in the comoving synchronous gauge to that in the Poisson gauge[23, 25]. After the publication several misprints were found in Eqs. (4.6) - (4.8) of Paper[12], which should be taken into account for deriving expressions (2.10) - (2.13). Perturbations in the decaying mode were derived in Paper [12], but they are omitted here.

$$z_i^{(2)} = P'(1 + P'')C_i, \tag{2.21}$$

and

$$\chi_{ij}^{(2)} = \left[P + \frac{1}{2}(P')^2 \right] D_{ij} + \frac{3}{7} P^2 \Delta D_{ij} + \delta \chi_{ij}, \tag{2.22}$$

where C_i satisfies

$$\Delta C_i = \left[-\frac{200}{9} \Psi_{0,i} + \frac{1}{2}(F_{,l}F_{,l})_{,i} - F_{,i}\Delta F \right], \tag{2.23}$$

and $\delta\chi_{ij}$ in Eq.(2.22) satisfies

$$\left(\frac{\partial^2}{\partial \eta^2} + \frac{2a'}{a} \frac{\partial}{\partial \eta} - \Delta \right) \delta\chi_{ij} = \frac{3}{7} P^2 \Delta^2 D_{ij} + \frac{1}{7} \left[P - \frac{5}{2}(P')^2 \right] \Delta D_{ij}. \tag{2.24}$$

Ψ_0 and Θ_0 are defined as

$$\Delta\Psi_0 \equiv \frac{9}{200} \left[F_{,kl}F_{,kl} - (\Delta F)^2 \right], \quad \Delta\Theta_0 \equiv \Psi_0 - \frac{3}{100} F_{,l}F_{,l}, \tag{2.25}$$

and $Q(\eta)$ satisfies

$$Q'' + \frac{2a'}{a}Q' = -\left[P - \frac{5}{2}(P')^2 \right]. \tag{2.26}$$

The above second order solutions are consistent with those shown in Eqs.(4.12) - (4.15) in Paper[12], which was derived using a transformation from the comoving synchronous gauge to the Poisson gauge. Note that ΔD_{ij} and $\Delta^2 D_{ij}$ correspond to $\tilde{G}_i^j$ and G_i^j in Eq.(2.25) of Paper[12], as $\Delta D_{ij} = -\tilde{G}_i^j$ and $\Delta^2 D_{ij} = -G_i^j$. Eq.(2.17) of Paper[12] with misprints must be replaced by the above correct equation (2.26).

In the above solutions, the ratios of terms including Θ_0 to terms including $F_{,l}F_{,l}$ and Ψ_0 are of the order of κ^2, where $\kappa \equiv |\mathbf{x}|/\eta$, and $|\mathbf{x}|$ and η are the characteristic spatial scale of local perturbation and the horizon size. Therefore, the terms including Θ_0 are negligible for local perturbations that are sufficiently smaller than the horizon size. Thus for $\kappa \ll 1$, we have

$$\phi^{(2)} = \psi^{(2)} = \zeta_1 \ F_{,l}F_{,l} + \zeta_2 \ 100\Psi_0, \tag{2.27}$$

$$z_i^{(2)} = P'(1 + P'')C_i, \tag{2.28}$$

and

$$\chi_{ij}^{(2)} = \frac{3}{7} P^2 \Delta D_{ij}. \tag{2.29}$$

III.   COSMOLOGICAL NEWTONIAN APPROXIMATION

In the cosmological Newtonian approximation, we assume that $\epsilon \equiv a|\mathbf{v}|/c \ll 1$ and $\kappa \equiv |\mathbf{x}|/\eta \ll 1$ but $\chi \equiv (\rho/\rho_B - 1)^{1/2}$ is arbitrary, and consider only ψ and $\phi(= \psi)$ as the metric perturbations[26]. Then, from the difference between the perturbed equation $R_0^0 = T_0^0 - \frac{1}{2}T_\mu^\mu$ and the background counterpart, we obtain

$$\Delta\psi = \frac{1}{2}a^2(\rho - \rho_B). \tag{3.1}$$

From the conservation equation $T_{;\nu}^{\mu\nu} = 0$ and the energy-momentum tensor (for the perfect fluid with pressure p) $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$, we obtain the equation of continuity

$$\rho' + \frac{3a'}{a}\rho + (\rho v^i)_{,i} = 0 \tag{3.2}$$

and the equation of motion

$$v^{i'} + v_{,j}^i v^j + \frac{a'}{a}v^i - \psi_{,i} + p_{,i}/\rho = 0, \tag{3.3}$$

where $v^i = au^i$. By solving these equations (3.1) -(3.3), we have ρ, v^i and ψ, which can determine the lowest-order RS effect in the form of spatial integration of $\psi_{,i}$ along a light path.

In what follows, we only consider the perturbative case with $\epsilon \ll 1$, $\kappa \ll 1$, and $\chi \ll 1$. First, expressing the perturbations as

$$\psi = \psi^{(1)} + \frac{1}{2}\psi^{(2)}, \qquad \rho = \rho_B + \delta\rho^{(1)} + \frac{1}{2}\delta\rho^{(2)}, \qquad v^i = v^{(1)\,i} + \frac{1}{2}v^{(2)\,i}, \tag{3.4}$$

the first order equations for these perturbations in the pressureless case are given by

$$\begin{aligned}
\Delta\psi^{(1)} &- \frac{1}{2}a^2\delta\rho^{(1)} = 0,\\
\delta\rho^{(1)\prime} &+ \frac{3a'}{a}\delta\rho^{(1)} + \rho_B v^{(1)\,i}{}_{,i} = 0,\\
v^{(1)\,i\prime} &+ \frac{a'}{a}v^{(1)\,i} + \psi^{(1)}_{,i} = 0,
\end{aligned} \tag{3.5}$$

and the first order solutions are

$$\delta\rho^{(1)}/\rho_B = \frac{1}{\rho_B a^2}[(a'/a)P' - 1]\Delta F, \tag{3.6}$$

$$v^{(1)i} = \frac{1}{2}P'F_{,i}, \tag{3.7}$$

and

$$\psi^{(1)} = -\frac{1}{2}\left(1 - \frac{a'}{a}P'\right)F, \tag{3.8}$$

where the above $\delta\rho^{(1)}, v^{(1)i}$ and $\psi^{(1)}$ are equal to Eqs.(2.12), (2.13), and (2.10) in the limit of $\kappa \ll 1$, respectively.

Next the corresponding Newtonian second order equations are

$$\begin{aligned}
\Delta\psi^{(2)} &- \frac{1}{2}a^2\delta\rho^{(2)} = 0,\\
\delta\rho^{(2)\prime} &+ \frac{3a'}{a}\delta\rho^{(2)} + 2[\delta\rho^{(1)}v^{(1)\,i}]_{,i} + \rho_B v^{(2)\,i}{}_{,i} = 0,\\
v^{(2)\,i\prime} &+ 2\,v^{(1)\,i}{}_{,j}v^{(1)\,j} + \frac{a'}{a}v^{(2)\,i} + \psi^{(2)}_{,i} = 0.
\end{aligned} \tag{3.9}$$

By substituting the above first order solutions Eqs. (3.6), (3.7), and (3.8) to Eq.(3.9) and eliminating $\psi^{(2)}$ and $v^{(2)\,i}$, we obtain

$$(\delta\rho^{(2)}/\rho_B)'' + \frac{a'}{a}(\delta\rho^{(2)}/\rho_B)' - \frac{1}{2}a^2\delta\rho^{(2)} = \frac{1}{4}(P')^2\Delta(F_{,l}F_{,l}) - \frac{1}{\rho_B a^4}[a^2 P'(\frac{a'}{a}P' - 1)]'(\Delta F F_{,i})_{,i}. \tag{3.10}$$

Assuming that $\psi^{(2)}$ is given by Eq.(2.27) and using the first line of Eq.(3.9), we can express $\delta\rho^{(2)}$ as

$$\delta\rho^{(2)}/\rho_B = \frac{2}{\rho_B a^2}\{\zeta_1\Delta(F_{,l}F_{,l}) + \frac{9}{2}\zeta_2[F_{,ij}F_{,ij} - (\Delta F)^2]\}. \tag{3.11}$$

By substituting this $\delta\rho^{(2)}/\rho_B$ to Eq.(3.10), we find that it is a solution of Eq.(3.10), so that $\psi^{(2)}$ given by Eq.(2.27) is also a solution of Eq.(3.9). Thus we proved that in the cosmological Newtonian limit with $\epsilon \ll 1$, $\kappa \ll 1$, and $\chi \ll 1$, the general relativistic second order solution is consistent with the Newtonian one.

IV. TEMPERATURE ANISOTROPY

In this section we consider the observed temperature of the CMB radiation which was emitted at the recombination epoch and received at the present epoch. The relation between the emitted and received temperatures $T_e \; T_o$ is

$$T_o = (\omega_o/\omega_e)T_e, \tag{4.1}$$

FIG. 1: Time dependent terms ζ_1' and $-\zeta_2'$ in second order temperature fluctuations as a function of redshift z. Ω_{m0} denotes the present value for the matter density parameter.
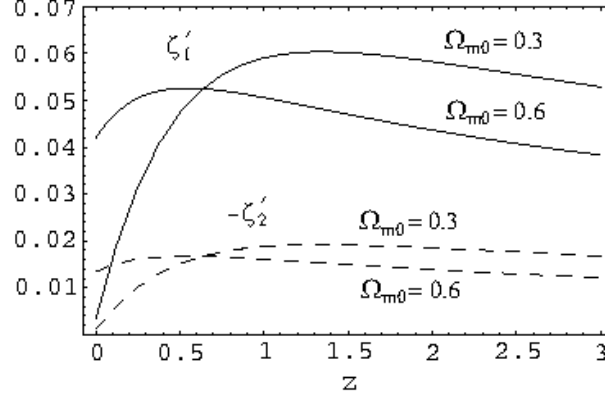
where $\omega = -g_{\mu\nu}u^\mu k^\nu$, u^μ is the observer's and emitter's velocities, and k^μ ($= dx^\mu/d\lambda$) is the wave vector of photons with affine parameter λ, which satisfies the null geodesic equation in the perturbed FRW universe. Solving this equation, the first order and second order perturbations of observed temperature $\Delta T^{(1)}$ and $\Delta T^{(2)}$ were derived in the gauge-invariant manner by Mollerach and Materrese[22]. When the background null geodesic rays are expressed as $x^{(0)\mu} = (\lambda, (\lambda_o - \lambda)e^i)$ and $k^{(0)\mu} = (1, -e^i)$, the first order temperature fluctuation is

$$\Delta T^{(1)}/T = \psi_e^{(1)} - \psi_o^{(1)} + [v_o^{(1)i} - v_e^{(1)i}]e_i + \tau + I_{1e}, \tag{4.2}$$

where

$$I_{1e} \equiv -\int_{\lambda_o}^{\lambda_e} d\lambda A^{(1)'},$$
$$A^{(1)} \equiv \psi^{(1)} + \phi^{(1)} + z_i^{(1)}e^i - \frac{1}{2}\chi_{ij}^{(1)}e^i e^j, \tag{4.3}$$

e^i is the unit (three-dimensional) directional vector, the subscripts e and o denote the epochs of emission and observation, τ is the temperature fluctuation at the emission epoch, and it is assumed that $x^{(1)\mu}(\lambda_o) = 0$ and $k^{(1)i}(\lambda_o) = 0$. The integral term I_{1e} represents the contribution due to the ISW effect.

Similarly, second order temperature fluctuation is given by

$$\Delta T^{(2)}/T = I_{2e} + [I_{1e}]^2 + (\Delta T^{(2)}/T)_{oe},$$
$$I_{2e} = -\frac{1}{2}\int_{\lambda_o}^{\lambda_e} d\lambda A^{(2)'} \tag{4.4}$$

where $(\Delta T^{(2)}T/T)_{oe}$ is the sum of terms consisting of second order and products of first order quantities at observer's and emitter's positions, and

$$A^{(2)} \equiv \psi^{(2)} + \phi^{(2)} + z_i^{(2)}e^i - \frac{1}{2}\chi_{ij}^{(2)}e^i e^j. \tag{4.5}$$

For quasi-linear perturbations with $\epsilon_m = \mathcal{O}(0.1)$, the contribution due to the RS effect can be written as $(\Delta T/T)_{\text{RS}} = I_{1e} + I_{2e}$. For scalar-type linear or quasi-linear perturbations, the terms $z_i^{(2)}$ and $\chi_{ij}^{(2)}$ in Eq.(4.5) can be neglected. Therefore, the second order contribution I_{2e} to the RS effect can be evaluated using only perturbations $\phi^{(2)'}$ and $\psi^{(2)'}$ which are obtained from Eq. (2.27) with the following conformal time derivatives

$$\zeta_1' = \frac{1}{4}\left\{P' - \frac{a'}{a}[(P')^2 + P] + \left[2\left(\frac{a'}{a}\right)^2 - \left(\frac{a'}{a}\right)'\right]PP'\right\}, \tag{4.6}$$

$$\zeta_2' = \frac{1}{18}\frac{a'}{a}P - \frac{1}{9}P' + \frac{1}{21}\left[\left(\frac{a'}{a}\right)' - 2\left(\frac{a'}{a}\right)^2\right]\left(PP' - \frac{1}{6}Q'\right) + \frac{5}{36}\frac{a'}{a}(P')^2, \tag{4.7}$$

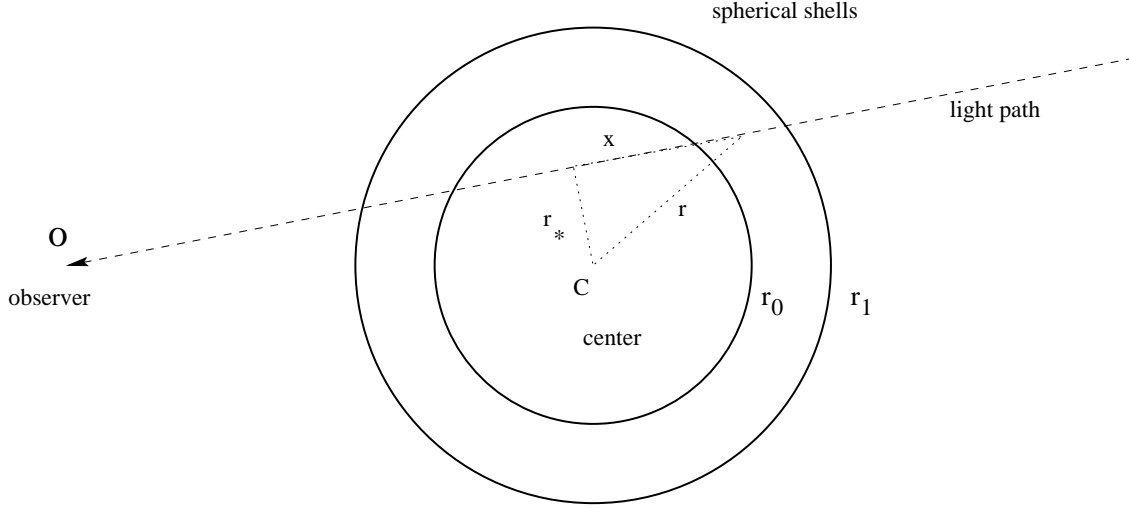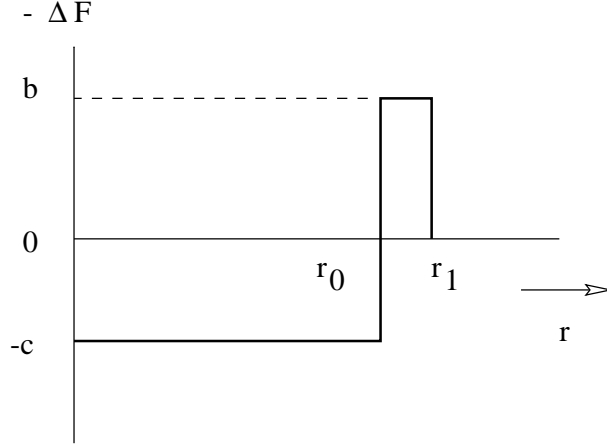FIG. 2: A light path in the spherical model.

FIG. 3: The matter density contrast for a top-hat type spherical void.

where $z_i^{(2)}$ and $\chi_{ij}^{(2)}$ are neglected. For perturbations sufficiently smaller than the Hubble scale, i.e., $\kappa \equiv |\mathbf{x}|/\eta \ll 1$, the dominant terms in $\phi^{(2)'}$ and $\psi^{(2)'}$ are those multiplied by $F_{,l}F_{,l}$ and Ψ_0.

For the Einstein-de Sitter (EdS) model ($\Lambda = 0$), we have $a \propto \eta^2, P = \eta^2/10$ and $Q = 0$, so that

$$\zeta_1 = \frac{3}{200}\eta^2, \quad \zeta_2 = -\frac{1}{210}\eta^2, \text{ and } \zeta_1'/\zeta_2' = -\frac{63}{20}. \tag{4.8}$$

For a Λ-dominated model with $(\Omega_{m0}, \Omega_{\Lambda 0} = (0.27, 0.73)$, numerical calculations of ζ_1 and ζ_2 show that ζ_1'/ζ_2' have the values $-2.68, -2.99, -3.08, -3.12, -3.14, -3.15$ for the redshifts $z = 0.05, 0.1, 0.2, 0.5, 1.0, 2.0$, respectively. For $z \gg 1$, ζ_1'/ζ_2' is nearly equal to $-63/20$ in the EdS model. As shown in Fig. 1, ζ_1' and ζ_2' have a peak well before the matter-Λ equality epoch $z_{m\Lambda} = ((1 - \Omega_{m0})/\Omega_{m0})^{1/3} - 1$, implying that the second order contribution to the RS effect in the quasi-linear regime is not so important at an accelerating epoch.

V.   BEHAVIORS OF FIRST ORDER AND SECOND ORDER TEMPERATURE FLUCTUATIONS IN A SPHERICAL TOP-HAT MODEL

In order to investigate the nature of the temperature fluctuations exerted by local inhomogeneities, we consider a simple toy model with a spherically symmetric density perturbation. In what follows, we assume that we are outside the local density perturbation (Fig.2). In this model, the potential function $F(\mathbf{x})$ can be written as a function of the comoving distance $r \equiv [\mathbf{x}^2]^{1/2}$ from the center of the perturbation as $F = F(r)$. Then, we have

$C_i = 0, D_{ij} = 0$ and $z_i^{(2)} = \chi_{ij}^{(2)} = 0$. For the top-hat type matter density perturbations, the functional form for $F(r)$ is given in terms of constant parameters b and c as

$$\Delta F = \frac{1}{r^2}\frac{d}{dr}(r^2 F_{,r}) = c, \quad -b, \quad 0 \tag{5.1}$$

for $0 \le r \le r_0$, $r_0 < r \le r_1$, $r_1 < r$, respectively(Fig. 3). This model represents a void if $c > 0$ and $b > 0$, or a lump if $c < 0$ and $b < 0$. Moreover, if $cr_0{}^3 = b(r_1{}^3 - r_0{}^3)$, the mass is totally compensating. $-\Delta F$ represents a value that is proportional to the matter density contrast in a comoving gauge. Note that $-F(\mathbf{x})$ describes the $\mathbf{x}$ dependence of the gravitational potential ψ. Here, r_0 and r_1 are the inner radius and the outer radius, respectively. $r_1 - r_0$ corresponds to the width of the wall.

Integrating Eq.(5.1), we obtain

$$F_{,r} = \frac{1}{r^2}\int_0^r dr r^2 \Delta F. \tag{5.2}$$

Then, under the condition that $F_{,r}$ is regular at the center $r = 0$, we have

$$F_{,r} = \frac{1}{3}cr, \quad \frac{1}{3}(c+b)r_0^3/r^2 - \frac{1}{3}br, \quad \frac{1}{3r^2}[cr_0^3 - b(r_1^3 - r_0^3)] \tag{5.3}$$

for $0 \le r \le r_0, r_0 < r \le r_1, r_1 < r$, respectively. From Eq.(2.25), on the other hand, we have

$$-\frac{100}{9}(r^2\Psi_{0,r})_{,r} = [r(F_{,r})^2]_{,r}, \tag{5.4}$$

so that

$$-\frac{100}{9}[\Psi_0(r)) - \Psi_0(0)] = \int_0^r \frac{(F_{,r})^2}{r}dr \equiv I. \tag{5.5}$$

Under the condition that Ψ_0 is regular at the center, $\Psi_0(0)$ is determined from the boundary condition that the perturbation is local, i.e., $\Psi_0 \to 0$ as $r \to \infty$. The explicit forms of I and $\Psi_0(0)$ are shown in Appendix C.

Integrating Eq.(5.3), we obtain

$$\begin{aligned}
F &= -\frac{1}{6}c\,(r^2 - r_0^2) + F_0 &\text{for} \quad 0 \le r < r_0 \\
&= -\frac{1}{3}(c+b)\,r_0^3\Big(\frac{1}{r} - \frac{1}{r_0}\Big) - \frac{1}{6}b\,(r^2 - r_0^2) + F_0 &\text{for} \quad r_0 \le r < r_1 \\
&= -\frac{1}{3r}[cr_0^3 - b(r_1^3 - r_0^3)] &\text{for} \quad r_1 \le r,
\end{aligned} \tag{5.6}$$

where

$$F_0 = \frac{1}{3}(c+b)\,r_0^3\Big(\frac{1}{r_1} - \frac{1}{r_0}\Big) + \frac{1}{6}b\,(r_1^2 - r_0^2) - \frac{1}{3r_1}[cr_0^3 - b(r_1^3 - r_0^3)]. \tag{5.7}$$

As shown in Fig. 4, the thinner wall (i.e.,$(r_1 - r_0)/r_1 \ll 1$) gives deeper potential for a lump provided that the matter density at the center is fixed (i.e., c is constant).

Using Eqs.(2.10), (4.2), and (4.3), the first order temperature fluctuation due to the local perturbation is given by the integral term in Eq. (4.2), which represents the ISW effect,

$$(\Delta T^{(1)}/T)_{loc} = I_{1e} = -2\int_{\lambda_o}^{\lambda_e} d\lambda \phi^{(1)\prime}, \tag{5.8}$$

where

$$\phi^{(1)\prime} = \frac{1}{2}\Big\{\frac{a'}{a} + \Big[\frac{a''}{a} - 3\Big(\frac{a'}{a}\Big)^2\Big]P'\Big\}F. \tag{5.9}$$

Now we consider a light path passing through the center of a spherical compensating void/lump and assume that its outer radius r_1 is sufficiently small compared with the Hubble radius at the time the photons pass through the center. Then the integration of F derived in Appendix C reduces to

$$\int_0^\infty F dr = -\frac{1}{9}c(r_0)^3\,(1+y)\ln(1+1/y) \tag{5.10}$$

FIG. 4: The potential function $-F(r)$ for a top-hat type compensated lump with $c = -0.1$ and $r_1 = 0.1$.
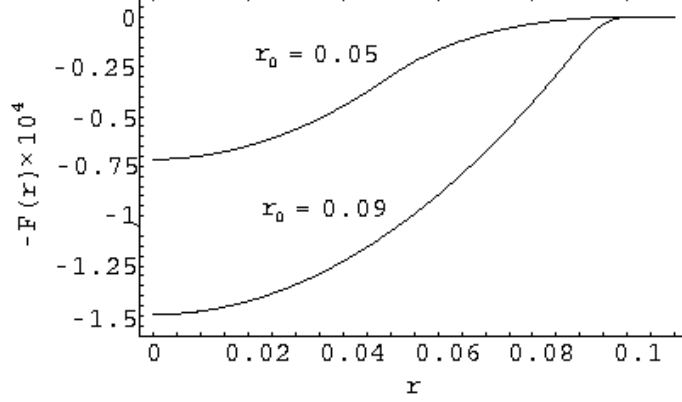
where $y = b/c$ and $r_1/r_0 = (1 + 1/y)^{1/3}$. From the above equations, we can express the first order temperature fluctuation as

$$(\Delta T^{(1)}/T)_{loc} \approx \frac{2}{9} \, c \, (r_1)^3 \, w_1(y) \Big[\frac{a'}{a} + \Big(\frac{a''}{a} - 3\Big(\frac{a'}{a}\Big)^2\Big) P'\Big]\Big|_{\eta = \eta_c}, \tag{5.11}$$

where $w_1(y) = -y \ln(1 + 1/y)$ which is a negative definite function for $y > 0$ and $\eta_0 - \eta_c$ represents the comoving distance to the void/lump. Using Eqs.(2.16), (2.6), and (2.7), c can be written in terms of the matter density contrast $(\epsilon_m)_c$ in the comoving gauge at the center of the inhomogeneity as

$$c = (\epsilon_m)_c \, 2\Big[2\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{a}\Big]\Big[\frac{a'}{a}P' - 1\Big]^{-1}. \tag{5.12}$$

From Eq.(5.11) and Eq.(5.12), we find that the first order temperature fluctuation due to a compensated void or lump is approximately written as $\mathcal{O}[(\Delta T^{(1)}/T)_{loc}] \sim (\epsilon_m)_c r_1^3$. As shown in Figs. 5 and 6 a compensated spherical void redshifts the photons whereas a compensated spherical lump blueshifts the photons at first order irrespective of the width of the wall.

In a similar manner, we can express the second order temperature fluctuation when the photons pass through the center of a compensating void/lump $(= -\frac{1}{2}\int_{\lambda_o}^{\lambda_e} d\lambda A^{(2)'})$ as

$$(\Delta T^{(2)}/T)_{loc} \approx \frac{4}{27} \, c^2 \, (r_1)^3 \, w_2(y)(\zeta_1 + 9 \, \zeta_2)'|_{\eta = \eta_c}, \tag{5.13}$$

$$w_2(y) \equiv y[1 - y \ln(1 + 1/y)], \tag{5.14}$$

where c is given by Eq.(5.12). Note that $w_2(y)$ is a positive definite function for $y > 0$. The order of the second order temperature fluctuation is $\mathcal{O}[(\Delta T^{(2)}/T)_{loc}] \sim -(\epsilon_m)_c^2 r_1^3$. As shown in Figs. 5 and 6, either type (void or lump) of density perturbation redshifts the photons irrespective of the width of the wall. This behavior suggests that the second order gravitational effect leads to a flow of matter from the wall to inside the wall in either case. Then the gravitational potential becomes smaller and photons passing through the center of the void(lump) get further redshifts.

To see this quantitatively, we calculate the ratio U of the second order temperature fluctuation to the first order one for a spherical void(lump) at redshift z,

$$U \equiv (\Delta T^{(2)}/T)_{loc}(z)/(\Delta T^{(1)}/T)_{loc}(z), \tag{5.15}$$

which is proportional to c. Plugging Eq.(5.12) into Eq.(5.15), we have

$$U = \frac{w_2(y)}{w_1(y)} \frac{\Big[2\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{a}\Big] (\zeta_1 + 9\zeta_2)'}{6\Big(\frac{a'}{a}P' - 1\Big)\Big\{\frac{a'}{a} + \Big[\frac{a''}{a} - 3\Big(\frac{a'}{a}\Big)^2\Big] P'\Big\}}(\epsilon_m)_c. \tag{5.16}$$

FIG. 5: The first and second order temperature fluctuations as functions of the matter density parameter at present Ω_{m0} for photons passing through the center of a compensated spherical void at $z \sim 0$. The matter density contrast at the center is $(\epsilon_m)_c = -0.3$ and $r_0 = 0.09 H_0^{-1}$, $r_1 = 0.1 H_0^{-1}$, where H_0 is the Hubble constant. The dashed and dashed-dotted curves denote $(\Delta T^{(1)}/T)_{loc}$ and $(\Delta T^{(2)}/T)_{loc}$, respectively. The solid curve represents the total temperature fluctuation $(\Delta T^{(1)}/T)_{loc} + (\Delta T^{(2)}/T)_{loc}$.
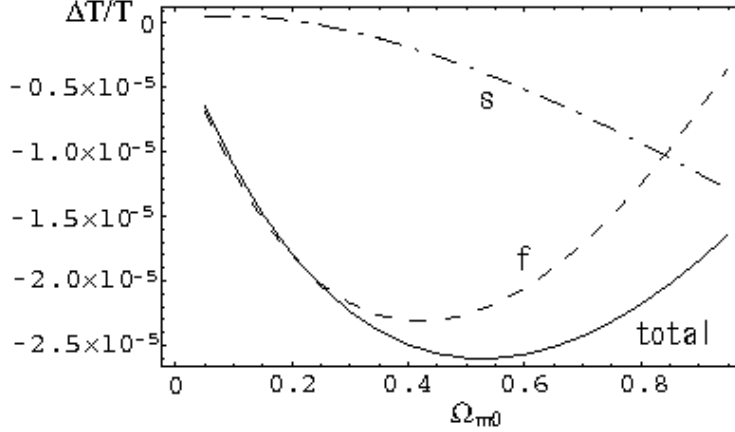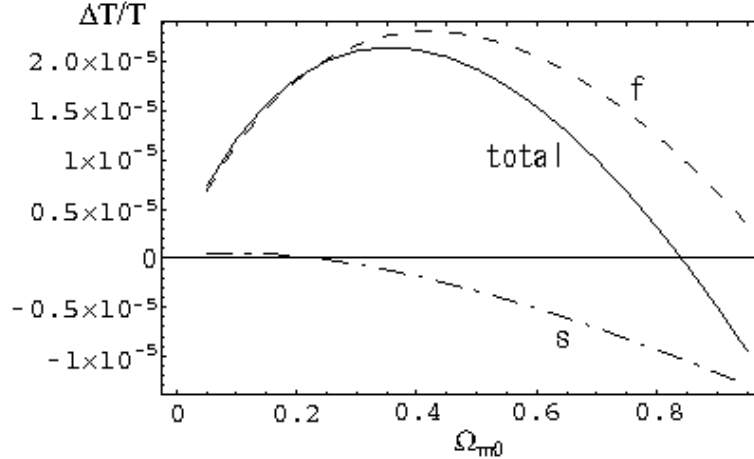

FIG. 6: For a compensated spherical lump. The parameters are as the same as in Fig. 5. The matter density contrast is $(\epsilon_m)_c = 0.3$.

As shown in Fig. 7, the absolute value $|U|$ for a fixed value $(\epsilon_m)_c$ monotonically increases as the redshift z of a void/lump increases. This behavior is naturally expected, since all the models approach to the EdS model for which U is infinite (because of $\phi^{(1)'} = 0$) in the limit $z \to \infty$. It turns out that the ratio U is positive (negative) for $(\epsilon_m)_c < 0 (> 0)$, respectively. In other words, we have positive (negative) correlation for a void (lump) between the first and the second order fluctuations. Thus, we conclude that *the first order temperature fluctuation corresponding to the linear integrated Sachs-Wolfe (ISW) effect is enhanced(suppressed) by the second order effect for a void(lump)*.

For $(r_1 - r_0)/r_1 = 0.2$, or equivalently, $y = b/c = 1.049$, the behaviors of $(\Delta T^{(1)}/T)_{loc}(z)/(\Delta T^{(1)}/T)_{loc}(0)$ and $0.05 (\Delta T^{(2)}/T)_{loc}(z)/(\Delta T^{(2)}/T)_{loc}(0)$ are shown in Fig. 8, when c is fixed. In contrast to the first order fluctuations which decrease as z increases, it is found that the second order fluctuations have a peak at a certain epoch z_p. For a model with $(\Omega_{m0}, \Omega_{\Lambda 0}) = (0.27, 0.73)$, we have $z_p \sim 1$.

VI.   CONCLUDING REMARKS

In this paper we have confirmed that in the nonzero-Λ cosmological model, the second order solutions (in the Poisson gauge) of scalar-type perturbations in general relativistic theory coincide with the solutions in

FIG. 7: The ratio between the second and first order temperature fluctuations U for a light path passing through the center as a function of the redshift z of the void/lump. The solid, dashed, and dashed-dotted curves correspond to flat models with $\Omega_{m0} = 0.9, 0.6$, and 0.3, respectively. The width of the wall is chosen to be 20% of the outer radius, i.e., $(r_1 - r_0)/r_1 = 0.2$. It turns out that the dependence on the ratio between the width and the outer radius is not prominent.
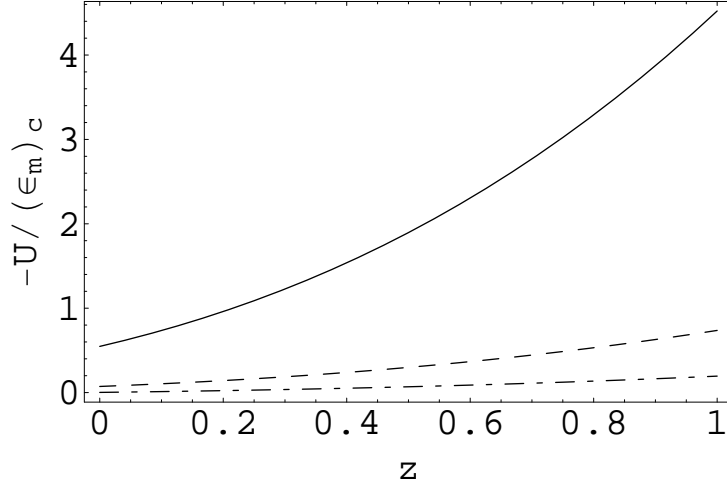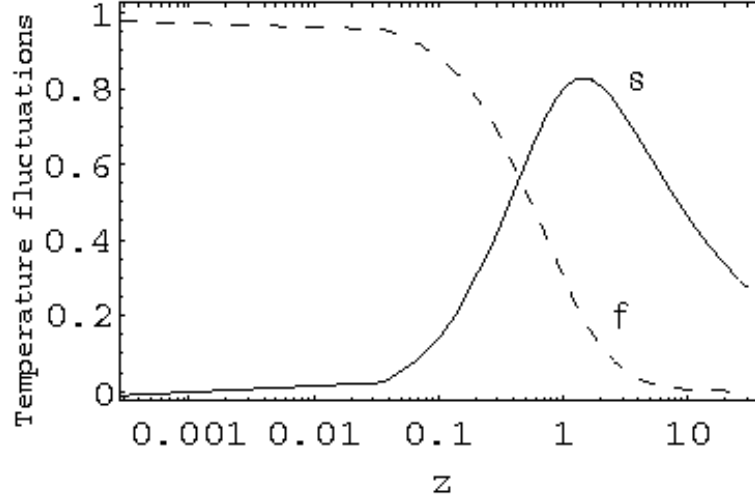

FIG. 8: The first order and second order temperature fluctuations as a function of the redshift z of a void/lump with wall width $(r_1 - r_0)/r_1 = 0.2$, a density contrast parameter c, and density parameters $(\Omega_{m0}, \Omega_{\Lambda 0}) = (0.27, 0.73)$. The solid and dashed curves denote $0.05 \, (\Delta T^{(2)}/T)_{loc}(z)/(\Delta T^{(2)}/T)_{loc}(0)$ and $(\Delta T^{(1)}/T)_{loc}(z)/(\Delta T^{(1)}/T)_{loc}(0)$, respectively.

the cosmological Newtonian theory in the 'Newtonian' limit, provided that the size of the perturbation is sufficiently small compared with the horizon scale. Thus, the second order temperature fluctuation due to a local perturbation sufficiently smaller than the horizon scale can be calculated using the Newtonian approximation as long as the solutions satisfying the Poisson equation and equations of continuity and motion are used in the Newtonian approximation.

In order to clarify the behavior of second order temperature fluctuations, we considered a simple spherical top-hat type local density perturbation (void/lump) in which the pressureless matter is totally compensating and whose size is smaller than the horizon size. We have found that the first order temperature fluctuation corresponding to the linear integrated Sachs-Wolfe (ISW) effect is enhanced(suppressed) by the second order effect for a compensated void(lump). As a function of redshift z of the local perturbation, the amplitude of the second order temperature fluctuation due to a void(lump) has a peak at an epoch $z \sim 1$ for a fixed density contrast and the comoving size, whereas the amplitude of the first order fluctuation increases monotonically. This means that the second order temperature fluctuations may play an important role especially for quasi-

linear objects such as voids with radius $r = 100 - 200 h^{-1}$Mpc at $z \sim 1$, which may lead to some observable imprints at an angle $\sim 5°$ in the sky ([10], [11], [27]).

APPENDIX A: THE MAIN COMPONENTS OF FIRST ORDER AND SECOND ORDER EINSTEIN EQUATIONS

The first order and second order components of Ricci and Einstein tensors are expressed using the symbols $\underset{1}{\delta}$ and $\underset{2}{\delta}$ as $R_\mu^\nu = (R_\mu^\nu)_B + \underset{1}{\delta} R_\mu^\nu + \underset{2}{\delta} R_\mu^\nu$ and $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R_\gamma^\gamma$. The energy-momentum tensor is $T_\mu^\nu = \rho u_\mu u^\nu$. In what follows, we assume the metric in (2.1) with conditions (2.8) and (2.9). For further details see ([28], [29], [30]).

For the first order perturbations, we obtain the components of Ricci and Einstein tensors

$$a^2 \underset{1}{\delta} R_0^0 = 6\left[\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right]\psi^{(1)} - \left[\Delta\psi^{(1)} + 3\frac{a'}{a}\psi^{(1)} + 3\phi^{(1)''} + 3\frac{a'}{a}\phi^{(1)}\right], \tag{A1}$$

$$a^2 \underset{1}{\delta} R_0^i = 2\phi_{,i}^{(1)'} + 2\frac{a'}{a}\psi_{,i}^{(1)}, \tag{A2}$$

$$\begin{aligned}
a^2 \underset{1}{\delta} G_i^j &= a^2[\underset{1}{\delta} R_i^j - \frac{1}{2}\delta_i^j \underset{1}{\delta} R_\mu^\mu] = \left\{2\frac{a'}{a}\phi^{(1)'} + \left[\frac{4a''}{a} - 2\left(\frac{a'}{a}\right)^2\right]\psi^{(1)} + \Delta\psi^{(1)}\right. \\
&+ \left. 2\phi^{(1)''} + \frac{4a'}{a}\phi^{(1)'} - \Delta\phi^{(1)}\right\}\delta_{ij} + \phi_{,ij}^{(1)} - \psi_{,ij}^{(1)}
\end{aligned} \tag{A3}$$

and the components of energy-momentum tensor are

$$a^2[\underset{1}{\delta} T_0^0 - \frac{1}{2}(\underset{1}{\delta} T_\mu^\mu)] = -\frac{1}{2}\rho_B a^2 \delta^{(1)}\rho/\rho_B, \tag{A4}$$

$$a^2 \underset{1}{\delta} T_0^i = -\rho_B a^2 v^{(1)i} \tag{A5}$$

$$a^2 \underset{1}{\delta} T_j^i = 0. \tag{A6}$$

For the second order perturbations, we obtain the following components

$$\begin{aligned}
a^2 \underset{2}{\delta} R_0^0 &= a^2 \underset{2}{\delta} \tilde{R}_0^0 + \psi_{,l}^{(1)}\psi_{,l}^{(1)} + 3\psi^{(1)'}\phi^{(1)'} - 2\phi^{(1)}\Delta\psi^{(1)} \\
&+ \phi_{,l}^{(1)}\psi_{,l}^{(1)} - 6\frac{a'}{a}\phi^{(1)}\phi^{(1)'} - 6\phi^{(1)}\phi^{(1)''} - 3(\phi^{(1)'})^2 \\
&+ 2\psi^{(1)}\left[\Delta\psi^{(1)} + 6\frac{a'}{a}\psi^{(1)'} + 3\phi^{(1)''} + 3\frac{a'}{a}\phi^{(1)'}\right] - 12\left[\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right](\psi^{(1)})^2,
\end{aligned} \tag{A7}$$

$$a^2 \underset{2}{\delta} R_0^i = a^2 \underset{2}{\delta} \tilde{R}_0^i - 4\frac{a'}{a}\psi^{(1)}\psi_{,i}^{(1)} + 4\frac{a'}{a}\phi^{(1)}\psi_{,i}^{(1)} - 2\phi^{(1)'}\psi_{,i}^{(1)} + 4\phi^{(1)'}\phi_{,i}^{(1)} + 8\phi^{(1)}\phi_{,i}^{(1)'}, \tag{A8}$$

$$\begin{aligned}
a^2 \underset{2}{\delta} G_i^j &= a^2 \underset{2}{\delta} \tilde{G}_j^i + W\delta_{ij} + \psi_{,i}^{(1)}\psi_{,j}^{(1)} + 2(\psi^{(1)} - \phi^{(1)})\psi_{,ij}^{(1)} \\
&- \phi_{,i}^{(1)}\psi_{,j}^{(1)} - \phi_{,j}^{(1)}\psi_{,i}^{(1)} + 3\phi_{,i}^{(1)}\phi_{,j}^{(1)} + 4\phi^{(1)}\phi_{,ij}^{(1)},
\end{aligned} \tag{A9}$$

where

$$\begin{aligned}
W &\equiv 4\left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right](\psi^{(1)})^2 - 8\frac{a'}{a}\left[\psi^{(1)}\psi^{(1)'} + \psi^{(1)}\phi^{(1)'} - \phi^{(1)}\phi^{(1)'}\right] + \left[\phi^{(1)} - 2\psi^{(1)}\right]'\phi^{(1)'} \\
&- \psi_{,l}^{(1)}\psi_{,l}^{(1)} - 2\phi_{,l}^{(1)}\phi_{,l}^{(1)} - 4\phi^{(1)}\Delta\phi^{(1)} + 2\left[\phi^{(1)} - \psi^{(1)}\right]\left[2\phi^{(1)''} + \Delta\psi^{(1)}\right],
\end{aligned} \tag{A10}$$

$$a^2 \underset{2}{\delta} \tilde{R}_0^0 = 3\Big[\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{a}\Big]\psi^{(2)} - \frac{1}{2}\Big[\Delta\psi^{(2)} + 3\frac{a'}{a}\psi^{(2)\prime} + 3\phi^{(2)\prime\prime} + 3\frac{a'}{a}\phi^{(2)\prime}\Big],$$

(A11)

$$a^2 \underset{2}{\delta} \tilde{R}_0^i = \phi^{(2)\prime}{}_{,i} + \frac{a'}{a}\psi^{(2)}_{,i} + \Big[2\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{a} - \frac{1}{4}\Delta\Big]z_i^{(2)},$$

(A12)

$$
\begin{aligned}
a^2 \underset{2}{\delta} \tilde{G}_i^j \; = \; & \Big\{ \frac{1}{2}\Delta\psi^{(2)} + \frac{a'}{a}\psi^{(2)\prime} + \Big[\frac{2a''}{a} - \Big(\frac{a'}{a}\Big)^2\Big]\psi^{(2)} + \phi^{(2)\prime\prime} \\
& + \frac{2a'}{a}\phi^{(2)\prime} - \frac{1}{2}\Delta\phi^{(2)}\Big\}\delta_{ij} + \frac{1}{2}[\phi^{(2)}_{,ij} - \psi^{(2)}_{,ij}] \\
& - \frac{a'}{2a}[z^{(2)}_{i,j} + z^{(2)}_{j,i}] - \frac{1}{4}[z^{(2)}_{i,j} + z^{(2)}_{j,i}]' + \frac{1}{4}\Big[\chi''_{ij}{}^{(2)} + \frac{2a'}{a}\chi'_{ij}{}^{(2)} - \Delta\chi_{ij}{}^{(2)}\Big].
\end{aligned}
$$

(A13)

The latter three components $a^2 \underset{2}{\delta} \tilde{R}_0^0$, $a^2 \underset{2}{\delta} \tilde{R}_0^i$, and $a^2 \underset{2}{\delta} \tilde{G}_i^j$ are those which do not include any second order terms consisting of products of first order quantities in $a^2 \underset{1}{\delta} R_0^0$, $a^2 \underset{1}{\delta} R_0^i$, and $a^2 \underset{1}{\delta} G_i^j$. For energy-momentum tensor, we have

$$a^2\Big[\underset{2}{\delta} T_0^0 - \frac{1}{2}\underset{2}{\delta} T_\mu^\mu\Big] = \rho_B a^2\Big[-\frac{1}{4}\delta^{(2)}\rho/\rho_B + \sum_i (v^{(1)i})^2\Big],$$

(A14)

$$a^2 \underset{2}{\delta} T_0^i = -\frac{1}{2}\rho_B a^2 v^{(2)i} - a^2(\delta^{(1)}\rho + 2\psi^{(1)}\rho_B)v^{(1)i} - a^2\rho_B \; v^{(1)0}v^{(1)i},$$

(A15)

$$a^2 \underset{2}{\delta} T_j^i = a^2\rho_B \; v^{(1)i}v^{(1)j}.$$

(A16)

APPENDIX B: SECOND ORDER SOLUTIONS OF EINSTEIN EQUATIONS

In this Appendix we show the solutions of second order Einstein equations. First, using Eq.(A9) in Appendix A and the expressions of first order perturbations, we obtain

$$
\begin{aligned}
& \Big\{ \frac{1}{2}\Delta\psi^{(2)} \; + \; \frac{a'}{a}\psi^{(2)\prime} + \Big[\frac{2a''}{a} - \Big(\frac{a'}{a}\Big)^2\Big]\psi^{(2)} + \phi^{(2)\prime\prime} + \frac{2a'}{a}\phi^{(2)\prime} - \frac{1}{2}\Delta\phi^{(2)}\Big\}\delta_{ij} + \frac{1}{2}[\phi^{(2)}_{,ij} - \psi^{(2)}_{,ij}] \\
& - \frac{a'}{2a}[z^{(2)}_{i,j} + z^{(2)}_{j,i}] - \frac{1}{4}[z^{(2)}_{i,j} + z^{(2)}_{j,i}]' + \frac{1}{4}\Big[\chi''_{ij} + \frac{2a'}{a}\chi'_{ij} - \Delta\chi_{ij}\Big] + W\delta_{ij} \\
& = \Big\{\Big[\Big(\frac{a'}{a}\Big)^2 - \frac{a''}{2a}\Big](P')^2 - \frac{1}{2}\Big(1 - \frac{a'}{a}P'\Big)^2\Big\}F_{,i}F_{,j} - \Big(1 - \frac{a'}{a}P'\Big)^2 FF_{,ij}.
\end{aligned}
$$

(B1)

Here $FF_{,ij} = \frac{1}{2}(F^2)_{,ij} - F_{,i}F_{,j}$ and $F_{,i}F_{,j}$ is divided into following independent terms

$$F_{,i}F_{,j} = A\delta_{ij} + B_{,ij} + C_{i,j} + C_{j,i} + D_{ij},$$

(B2)

where the following conditions are imposed on C_i and D_{ij}

$$\delta^{ij}C_{i,j} = 0, \;\; \delta^{ij}D_{ij} = 0 \text{ and } \delta^{jk}D_{ij,k} = 0.$$

(B3)

Functions A, B, C_i and D_{ij} are determined as follows

$$A = \frac{100}{9}\Psi_0,$$

(B4)

$$B = -\frac{100}{3}\Theta_0,$$

(B5)

$$\Delta C_i = \Big[\frac{200}{9}\Psi_0 - \frac{1}{2}F_{,l}F_{,l}\Big]_{,i} + F_{,i}\Delta F,$$

(B6)

$$D_{ij} = F_{,i}F_{,j} - \frac{100}{3}\Psi_0\delta_{ij} + \frac{100}{3}\Theta_{0,ij} - (C_{i,j} + C_{j,i}). \tag{B7}$$

Corresponding to the relation (B2), Eq.(B1) is divided into four parts

$$2\frac{a''}{a}\psi^{(2)} + \frac{a'}{a}\psi^{(2)'} + \frac{1}{2}\Delta\psi^{(2)} - \left(\frac{a'}{a}\right)^2\psi^{(2)} + \phi^{(2)''} + \frac{2a'}{a}\phi^{(2)'} - \frac{1}{2}\Delta\phi^{(2)}$$
$$= -\left\{\left[\left(\frac{a'}{a}\right)^2 - \frac{2a''}{a}\right]\left(1 - \frac{a'}{a}P'\right)^2 + \frac{2a'}{a}\left(1 - \frac{a'}{a}P'\right)\left(\frac{a'}{a}P'\right)' - \frac{1}{4}\left[\left(\frac{a'}{a}P'\right)'\right]^2\right\}F^2$$
$$+ \frac{1}{4}\left(1 - \frac{a'}{a}P'\right)^2[2\Delta(F^2) - F_{,l}F_{,l}] + \left\{\left[\left(\frac{a'}{a}\right)^2 - \frac{a''}{2a}\right](P')^2 + \frac{1}{2}\left(1 - \frac{a'}{a}P'\right)^2\right\}\frac{100}{9}\Psi_0, \tag{B8}$$

$$\psi^{(2)} - \phi^{(2)} = \left(1 - \frac{a'}{a}P'\right)^2 F^2 + \left\{1 - \frac{2a'}{a}P' + \left[3\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right](P')^2\right\}\frac{100}{3}\Theta_0, \tag{B9}$$

$$z_i^{(2)'} + \frac{2a'}{a}z_i^{(2)} = -2\left\{1 - \frac{2a'}{a}P' + \left[3\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right](P')^2\right\}C_i, \tag{B10}$$

$$\chi_{ij}^{(2)''} + \frac{2a'}{a}\chi_{ij}^{(2)'} - \Delta\chi_{ij}^{(2)} = \frac{1}{2}\left\{1 - \frac{2a'}{a}P' + \left[3\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right](P')^2\right\}D_{ij}. \tag{B11}$$

Eliminating $\psi^{(2)}$ from Eqs.(B8) and (B9), we obtain an equation for $\phi^{(2)}$

$$\phi^{(2)''} + \frac{3a'}{a}\phi^{(2)'} + \left[\frac{2a''}{a} - \left(\frac{a'}{a}\right)^2\right]\phi^{(2)}$$
$$= \frac{1}{4}\left[\left(\frac{a'}{a}\right)'P' + \frac{a'}{a}\left(1 - \frac{2a'}{a}P'\right)\right]^2 F^2 + \frac{1}{4}\left\{1 - \frac{2a'}{a}P' + \left[5\left(\frac{a'}{a}\right)^2 - \frac{2a''}{a}\right](P')^2\right\}F_{,l}F_{,l}$$
$$+ \left\{3\left(\frac{a'}{a}\right)^2 - \frac{2a''}{a} + \frac{a'}{a}\left[\frac{8a''}{a} - 14\left(\frac{a'}{a}\right)^2\right]P' + \left[2\left(\frac{a''}{a}\right)^2 + 19\left(\frac{a'}{a}\right)^4\right.\right.$$
$$\left.\left. - 14\left(\frac{a'}{a}\right)^2\frac{a''}{a}\right](P')^2\right\}\frac{100}{3}\Theta_0 - \left\{1 - \frac{2a'}{a}P' + \left[3\left(\frac{a'}{a}\right)^2 - \frac{a''}{a}\right](P')^2\right\}\frac{100}{9}\Psi_0. \tag{B12}$$

Solving this equation and using Eq.(B9), we get the expressions for $\phi^{(2)}$ and $\psi^{(2)}$ given in Eqs.(2.17) - (2.18).

APPENDIX C: THE EXPRESSION OF Ψ_0 AND THE INTEGRATION OF $(F_{,r})^2$, Ψ_0, AND F ALONG A LIGHT PATH

The term I in Eq.(5.5) is

$$\begin{aligned}
I &= \frac{1}{18}c^2r^2 \qquad \text{for} \quad 0 \le r \le r_0 \\
&= \frac{1}{12}r_0^2(c - 3b)(c + b) - \frac{1}{36}r^2[(c+b)^2(r_0/r)^6 - 8(c+b)b(r_0/r)^3 - 2b^2] \qquad \text{for} \quad r_0 < r \le r_1 \\
&= \frac{100}{9}\Psi_0(0) - \frac{1}{36}[cr_0^3 - b(r_1^3 - r_0^3)]^2 r^{-4} \qquad \text{for} \quad r_1 < r, \tag{C1}
\end{aligned}$$

where

$$\begin{aligned}
\frac{100}{9}\Psi_0(0) &\equiv \frac{1}{12}r_0^2(c - 3b)(c + b) - \frac{1}{36}r_1^2[(c+b)^2(r_0/r_1)^6 - 8(c+b)b(r_0/r_1)^3 - 2b^2] \\
&- \frac{1}{36}[cr_0^3 - b(r_1^3 - r_0^3)]^2 r_1^{-4}. \tag{C2}
\end{aligned}$$

For $0 < r_* < r_0$, we obtain the following integrals by use of $u \equiv r/r_0$ and $y \equiv b/c$

$$\begin{aligned}
\int_0^\infty (F_{,r})^2 dx \;/\; (\frac{1}{18}c^2r_0^3) &= -4y(1+y)\ln\frac{u_1 + (u_1^2 - u_*^2)^{1/2}}{1 + (1 - u_*^2)^{1/2}} + \frac{2}{3}(1 - u_*^2)^{1/2}(1 - y^2)(1 + 2u_*^2) \\
&+ \frac{2}{3}(u_1^2 - u_*^2)^{1/2}y^2(u_1^2 + 2u_*^2) + (1+y)^2u_*^{-2}\left\{(u_1^2 - u_*^2)^{1/2}/u_1^2 - (1 - u_*^2)^{1/2}\right.
\end{aligned}$$

$$+ \; u_*^{-1}\tan^{-1}\frac{u_*[(u_1^2 - u_*^2)^{1/2} - (1 - u_*^2)^{1/2}]}{u_*^2 + [(u_1^2 - u_*^2)(1 - u_*^2)]^{1/2}}\Big\} + [1 + y(1 - u_1^3)]^2 u_*^{-2}[-(u_1^2 - u_*^2)^{1/2}/u_1^2$$

$$+ \; \frac{1}{u_*}\tan^{-1}\frac{u_*}{(u_1^2 - u_*^2)^{1/2}}] \tag{C3}$$

and

$$\frac{100}{9}\int_0^\infty \Psi_0 dx \;\; / \;\; (\frac{1}{18}c^2 r_0^3) = -4y(1+y)\ln\frac{u_1 + (u_1^2 - u_*^2)^{1/2}}{1 + (1 - u_*^2)^{1/2}} - \frac{3}{2}(1 - u_*^2)^{1/2}\Big[\frac{2}{9}(1 - y^2)(1 + 2u_*^2)$$

$$+ \; (3y - 1)(1 + y)\Big] - \frac{3}{2}(u_1^2 - u_*^2)^{1/2}\Big[\frac{2}{9}y^2(u_1^2 + 2u_*^2) - 2y(1+y)/u_1 - y^2 u_1^2\Big]$$

$$+ \; \frac{1}{4}(1+y)^2 u_*^{-2}\Big\{(u_1^2 - u_*^2)^{1/2}/u_1^2 - (1 - u_*^2)^{1/2}$$

$$+ \; u_*^{-1}\tan^{-1}\frac{u_*[(u_1^2 - u_*^2)^{1/2} - (1 - u_*^2)^{1/2}]}{u_*^2 + [(u_1^2 - u_*^2)(1 - u_*^2)]^{1/2}}\Big\} + \frac{1}{4}[1 + y(1 - u_1^3)]^2 u_*^{-2}\Big[-(u_1^2 - u_*^2)^{1/2}/u_1^2$$

$$+ \; \frac{1}{u_*}\tan^{-1}\frac{u_*}{(u_1^2 - u_*^2)^{1/2}}\Big], \tag{C4}$$

where $u_1 \equiv r_1/r_0, u_* \equiv r_*/r_0$ and $x \equiv (r^2 - r_*^2)^{1/2}$. For $r_* = 0$, we obtain the compensating case (i.e., $u_1^3 = 1 + 1/y$)

$$\int_0^\infty (F_{,r})^2 dr \;=\; \frac{100}{9}\int_0^\infty \Psi_0 dr$$

$$=\; \frac{2}{27}c^2 r_0^3 \;(1+y)(1 - 3y\ln u_1). \tag{C5}$$

The integration of F is derived as follows :

$$\int_0^\infty F dx = \int_{r_*}^{r_1} F\Big(r^2 - r_*^2\Big)^{-1/2} r dr \equiv c(r_0)^3 J\Big(u_*\Big), \tag{C6}$$

where

$$J\Big(u_*\Big) \;=\; -\frac{1}{9}\Big(1 - u_*^2\Big)^{1/2}(1+y)\Big(4 - u_*^2\Big) + \frac{1}{9}\Big(u_1^2 - u_*^2\Big)^{1/2}\Big[y\Big(u_1^2 - u_*^2\Big) + 3(1+y)/u_1\Big]$$

$$- \; \frac{1}{3}(1+y)\ln\frac{u_1 + \Big(u_1^2 - u_*^2\Big)^{1/2}}{1 + \Big(1 - u_*^2\Big)^{1/2}}. \tag{C7}$$

In the above derivation we assumed that the mass is totally compensating. For $r_* = 0$, we obtain

$$J(0) = -\frac{1}{3}(1+y)\ln\Big(u_1\Big) \;=\; -\frac{1}{9}(1+y)\ln(1 + 1/y). \tag{C8}$$

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